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1990 J. Phys. A: Math. Gen. 23 41

(http://iopscience.iop.org/0305-4470/23/1/011)

J. Phys. A: Math. Gen. 23 (1990) 41-51. Printed in the UK

# Non-point transformations in constrained theories

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Received 29 November 1988

**Abstract.** In the Lagrange formulation of a classical constrained dynamics the properties of non-point transformations (i.e. those depending not only on coordinates but also on their time derivatives) which result in physically equivalent theories are studied as well as their analogues in the corresponding Hamilton dynamics.

## 1. Introduction

In this paper the properties of a certain class of non-point transformations (or equivalently, of non-point changes of variables), which depend not only on coordinates but also on their time derivatives in the Lagrange formulation of a classical dynamics, as well as their analogues in the corresponding Hamilton formulation, are studied. A good choice of variables is often a powerful tool in the understanding of different features of the phenomena under consideration. So changes of variables are extensively used when solving different problems for a given theory. For example, in chiral theories the transition to normal coordinates is very useful, and when studying properties of Goldstone modes in theories with a spontaneous symmetry breaking the substitution of the type $\varphi = \rho \exp(i\theta)$ is considered, and so on. Symmetry transformations of a Langrangian, inculding gauge and supergauge transformations, may be also treated as changes of variables.

It is well known (see, for example, Goldstein 1957 or Gantmacher 1966) that for a point change of variables (of the type $q^i = f^i(q^I)$) being performed both in the equations of motion and in the action for $q$, we obtain for $q^I$ a theory which is physically equivalent to the original theory. To the point transformations in the Lagrange formalism there correspond certain canonical transformations in the Hamilton formalism. The situation differs when one performs a non-point transformation of the type

$$q^i = f^i(q^I, \dot{q}^I). \qquad (1.1)$$

Let us consider, for example, a non-constrained theory of variables $q$ described by the Lagrangian $L = L(q, \dot{q})$. The equations of motion for $q$ are second order with respect to the time derivatives. By substituting (1.1) into these equations one gets the third-order equations of motion for variables $q^I$. On the other hand, let us make the substitution (1.1) into the Lagrangian $L$ and consider the result

$$L^I(q^I, \dot{q}^I, \ddot{q}^I) \equiv L(f, \dot{f}) \qquad (1.2)$$

as the Lagrangian which describes a theory of variables $q'$. It is obvious that the equations of motion for $q'$ which correspond to the Lagrangian (1.2) are fourth order with respect to the time derivatives and, consequently, they cannot be equivalent both to the original second-order equations for $q$ and to the third-order equations for $q'$ obtained by direct substitution. Hence, we can conclude that a non-point change of variables leads in general to a physically non-equivalent theory. Certainly, the theory of variables $q'$ with the Lagrangian (1.2) has a sector which is equivalent to the original theory of variables $q$. It consists of the extremals whose initial data are confined to the condition (1.1) and to its first and second time derivatives (the second derivative $\ddot{q}$ should be expressed in terms of $q$ and $\dot{q}$ due to the original equations of motion). However, when considered by itself, the theory (1.2) is non-equivalent to (is 'broader' than) the original theory. In terms of Gitman and Tyutin (1986) the theory of variables $q'$ with the Lagrangian $L'$ (1.2) is the gauge of the original theory of variables $q$ with the Lagrangian $L(q, \dot{q})$.

The non-equivalence of the two theories for variables $q$ and $q'$ in the case of a non-point transformation is due to the fact that in the Lagrangian $L'$ there appeared time derivatives of a higher order than those in the Lagrangian $L$. However, these higher derivatives may enter the Lagrangian $L'$ in a combination which is a total time derivative. Equations of motion for variables $q'$ in such a case will also be second order, and one can suggest that the two theories, of variables $q$ and $q'$, are physically equivalent. It is the aim of this paper to prove that this suggestion is true.

So, we shall consider changes of variables of the form (1.1) which are restricted by the condition that the highest (the second) derivative in $L'(q', \dot{q}', \ddot{q}') = L(f, \dot{f})$ should appear only in a combination which is a total time derivative. In addition, we shall confine ourselves to infinitesimal transformations. Let us note that when Lagrangian symmetries are considered, infinitesimal transformations are usually sufficient, and their investigation is important in itself.

The paper is organised as follows. In section 2 non-constrained theories are examined. It is shown that if variables $q'$ satisfy Lagrange equations corresponding to the Lagrangian $L'$ (1.2), then variables $q$ which are connected with $q'$ by (1.1) satisfy Lagrange equations corresponding to the Lagrangian $L(q, \dot{q})$. It is also established that the transformations (1.1) is one-to-one on real trajectories (i.e. for $q(t)$ and $q'(t)$ satisfying proper Lagrange equations). In the proof, a transition to the Hamilton formulation is used. It is shown, as a by-product, that if two Lagrangians $L(q, \dot{q})$ and $L'(q', \dot{q}')$ are mutually connected, up to a total time derivative, by a transformation of the form (1.1) then the corresponding Hamilton formulations are connected by a canonical transformation. The complement is also true: if two Hamilton theories are connected by a canonical transformation then corresponding Lagrangians are connected, up to a total time derivative, by a certain, in general, non-point transformation. Thus, there exists a one-to-one correspondence between Lagrangian transformations, modulo a total time derivative, which are generated by a generally non-point change of variables, and canonical transformations of the corresponding Hamiltonian. In section 3 the consideration of section 2 is further generalised to theories with second-class constraints. Some additional restrictions are imposed on the change of variables (1.1). Namely, the number of primary as well as of all other constraints should be the same in the Hamilton formulation of the theory of variables $q'$ (for a more exact statement see section 3). In this case, too, the two theories of variables $q$ and variables $q'$, turn out to be physically equivalent and the transformation (1.1) is one-to-one on real trajectories. The Hamiltonians of the two theories are connected by a change of variables $p$, $q$, $\lambda$ ($\lambda$ are Lagrange multipliers to the primary constraints), and in the sector of variables $p$, $q$ this transformation is canonical (but it is dependent on $\lambda$). It is also shown that there exists a true canonical transformation of variables $p$ and $q$ which coincides with that mentioned above on the surface defined by the equations of primary constraints and by the equations $\lambda = \lambda(p, q)$, following from the Dirac method (Dirac 1964). This canonical transformation transfers the whole set of all the constraints of the theory of variables $q$ into the whole set of all the constraints of the theory of variables $q^I$.

It should be emphasised that the non-point transformation (1.1), which is one-to-one on real trajectories, is not of this sort for arbitrary trajectories, in contrast to the case of point transformations. As to the connection between Hamiltonians of the two theories, only the following properties of the transformation (1.1) are used in the proof: the highest time derivative appears in $L^I$ only through a total time derivative, and the number of primary constraints is conserved. Therefore the correspondence established between the Hamiltonians is also valid for any degenerate theory.

Finally, in section 4 the problem inverse to the problem of section 3 is considered: a change of variables is performed in the Hamiltonian formalism. It is shown that if the change obeys the following three conditions: (i) in the $p$, $q$ sector the change is canonical (dependent, perhaps, on $\lambda$), (ii) the $\lambda$ remain Lagrange multipliers in the transformed action (there are no time derivatives of $\lambda^I$), (iii) primary constraints of the new theory are independent of $\lambda^I$; then the corresponding Lagrangians are connected, up to a total time derivative, by a generally non-point change of variables $q$.

## 2. Non-point transformations in non-constrained dynamics

Let us consider a non-constrained system, described by the Lagrangian without the higher derivatives

$$L = L(q, \dot{q}) \qquad q = \{q^a\},\ a = 1, 2, \ldots, n.$$

Let us make an infinitesimal non-point change of variables

$$q^a = q^{Ia} + \Delta^a(q^I, \dot{q}^I). \tag{2.1}$$

Then we have for the Lagrangian $L(q, \dot{q})$ in the new variables†

$$L(q, \dot{q}) = L(q^I, \dot{q}^I) + \frac{\partial L(q^I, \dot{q}^I)}{\partial q^{Ia}} \Delta^a(q^I, \dot{q}^I) + \frac{\partial L(q^I, \dot{q}^I)}{\partial \dot{q}^{Ib}} \left( \frac{\partial \Delta^b(q^I, \dot{q}^I)}{\partial q^{Ia}} \dot{q}^{Ia} + \frac{\partial \Delta^b(q^I, \dot{q}^I)}{\partial \dot{q}^{Ia}} \ddot{q}^{Ia} \right). \tag{2.2}$$

We shall assume that the equations of motion for variables $q^I$ are second order with respect to the time derivatives. As was noted in section 1, this condition is necessary for the physical equivalence of two non-constrained theories connected by the change (2.1). It follows from the preservation of the order of the equations of motion for $q^I$ that $\ddot{q}^I$ appears in (2.2) as a total time derivative term. It means, in its turn, that there exists the function $F(q, \dot{q})$ such that

$$\frac{\partial F}{\partial \dot{q}^a} = \frac{\partial L}{\partial \dot{q}^b} \frac{\partial \Delta^b}{\partial \dot{q}^a}. \tag{2.3}$$

† Here and throughout the text, summation over repeated indices is understood. The indices are omitted in those cases when no misunderstanding can ensue. Remember that all the equalities are true up to the second-order terms in $\Delta$.

Equations (2.3) can be solved. Let us introduce the notation

$$\bar{\Delta}(p, q) = \Delta(q, \dot{q}(p, q)) \qquad \bar{F}(p, q) = F(q, \dot{q}(p, q))$$

where $\dot{q}(p, q)$ is the solution of the equations

$$p_a = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^a}$$

with respect to $\dot{q}$ (note that $\Delta(q, \dot{q}) = \bar{\Delta}(\partial L/\partial \dot{q}, q)$, and so on). Then the solution of equations (2.3) can be represented in the form

$$\bar{\Delta}^a(p, q) = \frac{\partial \varphi(p, q)}{\partial p_a} \qquad \bar{F}(p, q) = p_a \frac{\partial \varphi(p, q)}{\partial p_a} - \varphi(p, q)$$

where $\varphi$ is an arbitrary function of $p$ and $q$. The relation (2.2) may be rewritten as

$$L(q, \dot{q}) = L^I(q^I, \dot{q}^I) + \frac{\mathrm{d}F}{\mathrm{d}t} \tag{2.4}$$

$$L^I(q, \dot{q}) = L(q, \dot{q}) + \Delta L(q, \dot{q})$$

$$\Delta L = \frac{\partial L}{\partial q^a} \Delta^a + \left( \frac{\partial L}{\partial \dot{q}^b} \frac{\partial \Delta^b}{\partial q^a} - \frac{\partial F}{\partial q^a} \right) \dot{q}^a = \left( \frac{\partial L}{\partial q^a} \frac{\partial \varphi}{\partial p_a} + \frac{\partial \varphi}{\partial q^a} \dot{q}^a \right) \Bigg|_{p = \partial L/\partial \dot{q}} . \tag{2.5}$$

The two theories, whose Lagrangians differ in the total time derivative, are physically equivalent (Gitman and Tyutin 1986); in particular, their equations of motion coincide. Therefore, we shall use the Lagrangian $L^I(q^I, \dot{q}^I)$ for the theory of variables $q^I$. A question arises regarding whether the theories $(q; L)$ and $(q^I; L^I)$ are physically equivalent. As was noted in section 1, we shall answer this question in the affirmative.

Let us construct the Hamiltonians for the theories with the Lagrangians $L(q, \dot{q})$ and $L^I(q^I, \dot{q}^I)$

$$H(p, q) = (p\dot{q} - L(q, \dot{q}))|_{p = \partial L(q, \dot{q})/\partial \dot{q}}$$

$$H^I(p^I, \dot{q}^I) = (p^I \dot{q}^I - L^I(q^I, \dot{q}^I))|_{p^I = \partial L^I(q^I, \dot{q}^I)/\partial \dot{q}^I} .$$

Using the relation $\Delta H = -\Delta L$, which follows from the properties of the Legendre transformation, and (2.5), we obtain

$$H^I(p^I, q^I) = H(p^I, q^I) + \{H(p^I, q^I), \varphi(p^I, q^I)\} \tag{2.6}$$

$$p_a = p^I_a - \frac{\partial \varphi(p^I, q^I)}{\partial q^{Ia}} = p^I_a + \{p^I_a, \varphi(p^I, q^I)\} \tag{2.7}$$

$$q^a = q^{Ia} + \frac{\partial \varphi(p^I, q^I)}{\partial p^I_a} = q^{Ia} + \{q^{Ia}, \varphi(p^I, q^I)\}. \tag{2.8}$$

Thus, if the non-constrained Lagrangians $L$ and $L^I$ are connected by the change of variables (2.1), so that relation (2.4) is fulfilled, than the Hamilton formulations, of these theories are connected by the canonical transformation, and the transformation of the coordinates in the Lagrange and Hamilton formulations (2.1) and (2.8) coincide (after the identification $p^I = \partial L^I/\partial \dot{q}^I$ is made).

The reverse is also true. If the Hamilton formulations of the non-constrained systems are connected by the canonical transformation, then the corresponding Lagrange formulations are connected by a change of variables of the type (2.1) (so that the highest time derivative appears only as a total time derivative). Indeed, let the Hamiltonian $H(p, q)$ be given. Let us construct the corresponding Lagrangian

$$L(q, \dot{q}) = (p\dot{q} - H(p, q))|_{\dot{q}=\partial H/\partial p}.$$

Now let us make canonical transformation (2.7) and (2.8). In terms of the variables $p^I$, $q^I$ the theory is described by the Hamiltonian

$$H^I(p^I, q^I) = H(p, q) = H(p^I, q^I) + \Delta H(p^I, q^I)$$

$$\Delta H(p^I, q^I) = \{H(p^I, q^I), \varphi(p^I, q^I)\}.$$

The corresponding Lagrangian is

$$L^I(q^I, \dot{q}^I) = (p^I\dot{q}^I - H^I(p^I, q^I))|_{\dot{q}^I=\partial H^I/\partial p^I} = L(q^I, \dot{q}^I) + \Delta L.$$

Using the property of the Legendre transformation $\Delta L = -\Delta H$ we obtain

$$L^I(q^I, \dot{q}^I) = L(q, \dot{q}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^a}\frac{\partial \varphi}{\partial p_a} - \varphi\right)\bigg|_{p=\partial L/\partial \dot{q}}$$

which proves the reverse statement.

To prove that the theories $(q; L)$ and $(q^I; L^I)$ are physically equivalent we shall use the one-to-one correspondence between real trajectories in the Lagrangian and Hamiltonian formalisms, and the one-to-one correspondence between real trajectories of the two Hamilton theories connected by the canonical transformation. Let the sign $\Longleftrightarrow$ denote one-to-one correspondence between real trajectories. Then we can write

$$(q; L) \Longleftrightarrow (p, q; H) \Longleftrightarrow (p^I, q^I; H^I) \Longleftrightarrow (q^I; L^I).$$

Hence it follows that

$$(q; L) \Longleftrightarrow (q^I; L^I).$$

What are the transformations realising the correspondence between real trajectories $q$ and $q^I$? Let $(p^I, q^I)$ be a real trajectory in the theory $(p^I, q^I; H^I)$, where $q^I$ is a real trajectory of the theory $(q^I; L^I)$. Then $p$ and $q$, connected with $p^I$ and $q^I$ by relations (2.7) and (2.8), are real trajectories in the theory $(p, q; H)$, and $q$ is a real trajectory for the theory $(q; L)$. But on real trajectories (2.8) coincides with (2.1). hence, the change (2.1) transforms the real trajectory $q^I$ into the real trajectory $q$. The reverse transformation (on real trajectories) can be constructed in the following way: the change (2.7), (2.8) should be converted, and then, substituting $\partial L/\partial \dot{q}$ for the canonical momenta $p$, we shall obtain the change $q^I = q^I(q, \dot{q})$.

## 3. Non-point transformations in constrained dynamics

Here we shall consider non-point transformations for constrained theories. Let us confine ourselves to the theories with only second-class constraints in the Hamilton formulation. As to the change (2.1), we shall assume again that the highest time derivative enters the transformed Lagrangian as a total time derivative combination. We shall also impose additional restrictions on the properties of the change (2.1). Namely, we shall assume that (i) the ranks of the Hessian for two Lagrangians $L(q, \dot{q})$, $L^I(q^I, \dot{q}^I)$ are equal to each other, (ii) the total numbers of constraints in $L^I(q^I, \dot{q}^I)$ and $L(q, \dot{q})$ coincide one with another, (iii) the constraints and the Lagrange multipliers of the transformed theory are first-order perturbations with respect to $\Delta^a$ of the corresponding expressions of the starting theory. We shall show below that these conditions are sufficient for the physical equivalence of the two theories. But we do not know at present whether they are necessary. However, it is not difficult to give examples of the transformations in which some of the above-mentioned conditions are broken, and the transformed theory requires a greater amount of the initial data for the Cauchy problem, in comparison with the starting theory.

So, let us consider the constrained theory described by the Lagrangian $L(q, \dot{q})$. Let us divide the variables into two groups $q^a = (x^\alpha, X^i)$, $p_a = (\pi_\alpha, \Pi_i)$ so that the system of equations

$$\Pi_i = \frac{\partial L}{\partial \dot{X}^i}$$

allows us to solve the velocities $\dot{X}^i$ in terms of the coordinates $q^\alpha$, to momenta $\Pi_i$ and the Lagrange multipliers $\lambda^\alpha \equiv \dot{x}^\alpha$

$$\dot{X}^i = V^i(q, \Pi, \lambda).$$

Here

$$\left.\frac{\partial L}{\partial \dot{x}^\alpha}\right|_{\dot{x}=\lambda,\, \dot{X}=V} = f_\alpha(q, \Pi).$$

Further, we make the change of the variables (2.1) in the Lagrangian $L(q, \dot{q})$. So, equation (2.3), which ensures the absence of higher time derivatives in $L^I$, can be explicitly solved even in constrained theories. Namely, if we introduce the functions

$$\bar{\Delta}^a(q, \Pi, \lambda) = \Delta^a(q, \dot{q})|_{\dot{x}=\lambda,\, \dot{X}=V} \qquad \bar{F}(q, \Pi, \lambda) = F(q, \dot{q})|_{\dot{x}=\lambda,\, \dot{X}=V}$$

then

$$\bar{\Delta}^i = \frac{\partial \varphi(q, \Pi)}{\partial \Pi_i} - \bar{\Delta}^\alpha \frac{\partial f_\alpha(q, \Pi)}{\partial \Pi_i} \qquad \bar{F} = \Pi_i \bar{\Delta}^i + f_\alpha \bar{\Delta}^\alpha - \varphi$$

where $\varphi$ is an arbitrary function of $q$ and $\Pi$ (and does not depend on $\lambda$), $\bar{\Delta}^\alpha$ are arbitrary functions of $q$, $\Pi$, $\lambda$. Of course, the functions $\varphi$ and $\bar{\Delta}^\alpha$ must also ensure the correspondence between the constraints of the starting and the transformed theories.

For the Lagrangian $L^I$ we obtain

$$L^I(q^I, \dot{q}^I) = L(q^I, \dot{q}^I) + \Delta L(q^I, \dot{q}^I)$$

$$\Delta L(q, \dot{q}) = \frac{\partial L}{\partial q^a}\Delta^a + \dot{q}^a\left(\frac{\partial L}{\partial \dot{q}^b}\frac{\partial \Delta^b}{\partial q^a} - \frac{\partial F}{\partial q^a}\right)$$

$$= \left.\frac{\partial L}{\partial q^a}\Delta^a + \dot{q}^a\left(\frac{\partial \varphi}{\partial q^a} - \frac{\partial f_\alpha}{\partial q^a}\Delta^\alpha\right)\right|_{\lambda=\dot{x},\, \Pi=\partial L/\partial \dot{X}}.$$

It is convenient to use the Hamilton formalism to prove the physical equivalence of two theories with the Lagrangians $L(q, \dot{q})$ and $L^I(q^I, \dot{q}^I)$. Let us construct the Hamiltonian $H^{(I)}(p, q, \lambda)$ (Dirac 1964) for the theory with the Lagrangian $L(q, \dot{q})$ as

$$\begin{aligned} &H^{(1)}(p, q, \lambda) = (p\dot{q} - L(q, \dot{q}))|_{\dot{x}=\lambda,\, \dot{X}=V} \equiv H(\Pi, q) + \lambda^\alpha \phi_\alpha^{(1)}(p, q) \\ &H(\Pi, q) = (p\dot{q} - L(q, \dot{q}))|_{\pi_\alpha = f_\alpha,\, \dot{X}=V} \\ &\phi_\alpha^{(1)}(p, q) = \pi_\alpha - f_\alpha(\Pi, q) \end{aligned} \qquad (3.1)$$

where $\phi_\alpha^{(1)}$ are primary constraints. The Hamiltonian $H^{I(1)}(p^I, q^I, \lambda^I)$ for the theory with the Lagrangian $L^I(q^I, \dot{q}^I)$ is constructed in an analogous way. Its structure is defined by equations (3.1) with the replacement of all the non-primed quantities by primed ones (we assume here that the numbers of constraints $\phi^{(1)}$ and $\phi^{I(1)}$ are equal).

Using the relation between $L$ and $L^I$ we find

$$H^{I(1)}(p^I, q^I, \lambda^I) = H^{(1)}(p, q, \lambda)$$

$$p_a = p_a^I + \{p_a^I, W^{(1)}(p^I, q^I, \lambda^I)\} \equiv p_a^I + \nabla_{pa} \tag{3.2}$$

$$q^a = q^{Ia} + \{q^{Ia}, W^{(1)}(p^I, q^I, \lambda^I)\} \equiv q^{Ia} + \nabla_q^a \tag{3.3}$$

$$\lambda^\alpha = \lambda^{I\alpha} + \{H^{I(1)}(p^I, q^I, \lambda^I), \bar{\Delta}^\alpha(p^I, q^I, \lambda^I)\} \equiv \lambda^{I\alpha} + \nabla_\lambda^\alpha \tag{3.4}$$

where

$$W^{(1)}(p, q, \lambda) = p_a\bar{\Delta}^a - \bar{F} = \varphi(\Pi, q) + \phi_\alpha^{(1)}\bar{\Delta}^\alpha(p, q, \lambda).$$

Note that $\Delta^a$ and $\nabla_q^a$ are, in fact, identical transformations; namely, the following relation holds:

$$\nabla_q^a|_{\lambda = \dot{x},\, p = \partial L/\partial \dot{q}} = \Delta^a. \tag{3.5}$$

If we consider in the Lagrangian formalism, the change in momenta under the transformations (2.1) is the following:

$$\Delta p_a(q, \dot{q}) = \frac{\partial L^I(q^I, \dot{q}^I)}{\partial \dot{q}^{Ia}} - \frac{\partial L(q, \dot{q})}{\partial \dot{q}^a} = -\frac{\partial^2 L}{\partial \dot{q}^a \partial q^b}\Delta^b - \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}\dot{\Delta}^b + \frac{\partial \Delta L}{\partial \dot{q}^a}.$$

We can check that the relations

$$(\Delta p_a - \nabla p_a)|_{\lambda = \dot{x},\, p = \partial L/\partial q} = \left[ -\phi_\alpha^{(1)}\frac{\partial \bar{\Delta}^\alpha}{\partial q^a} + \{\phi_\alpha^{(1)}, H^{(1)}\}\frac{\partial \bar{\Delta}^\alpha}{\partial \dot{q}^a} - \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{X}^i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi}{\partial \Pi_i} - \left\{\frac{\partial \varphi}{\partial \Pi_i}, H^{(1)}\right\} - \bar{\Delta}^\alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_\alpha}{\partial \Pi_i} - \left\{\frac{\partial f_\alpha}{\partial \Pi_i}, H^{(1)}\right\}\right)\right)\right]_{\lambda = \dot{x},\, p = \partial L/\partial q} \tag{3.6}$$

are fulfilled.

Thus, we see that when two Lagrangians are connected by the non-point change of the variables, the corresponding Hamiltonians are also connected by some variable change which is canonical (but dependent on the parameters $\lambda^\alpha$) in the $p$, $q$ sector. Note here that this fact is true for theories both with second-class constraints and with first-class ones. Indeed, in proving it we have used only the coicidence of the primary constraints numbers in the theories $(q; L)$ and $(q^I; L^I)$.

It should be noted, however, that the change of variables $(p, q, \lambda) \to (p^I, q^I, \lambda^i)$ which transforms the Hamilton action $S(p, q, \lambda)$, corresponding to the Lagrangian $L(q, \dot{q})$, into the Hamilton action $S^I(p^I, q^I, \lambda^I)$, corresponding to the Lagrangian $L^I(q^I, \dot{q}^I)$, differs from (3.4) in the $\lambda$ sector. Namely, consider the Hamilton action $S(p, q, \lambda)$

$$S(p, q, \lambda) = \int \mathrm{d}t(p\dot{q} - H^{(1)}(p, q, \lambda))$$

and make the change of variables in it

$$p = p^I + \nabla_p \qquad q = q^I + \nabla_q \qquad \lambda = \lambda^I + \delta\lambda.$$

We have

$$S(p, q, \lambda)=\int \mathrm{d}t\left[p^I\dot{q}^I - H^{(1)}(p^I, q^I, \lambda^I)+\phi_\alpha^{I(1)}\left(\nabla_\lambda^\alpha + \dot{\lambda}^{I\beta}\frac{\partial\bar{\Delta}^\alpha}{\partial\lambda^{I\beta}} - \delta\lambda^\alpha\right)\right].$$

Hence it is clear that $\delta\lambda$ should be taken in the form

$$\delta\lambda^\alpha = \nabla_\lambda^\alpha + \dot{\lambda}^{I\beta}\frac{\partial\bar{\Delta}^\alpha}{\partial\lambda^{I\beta}}. \tag{3.7}$$

Thus, the change of variables, connecting two theories in the Hamiltonian formalism, is also a non-point one (we can now show that after the identification $\lambda=\dot{x}$, $\lambda^I=\dot{x}^I$, $p=\partial L/\partial\dot{q}$ that the transformation (3.7) coincides with the transformation connecting $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}x^I/\mathrm{d}t$). For non-constrained theories this transformation in the Hamiltonian formalism was a *point* transformation which, in fact, enabled us to prove the physical equivalence of the two theories.

We shall now construct a new *point* (strictly canonical in the $p$, $q$ sector) transformation which, however, coincides with (3.2), (3.3), (3.7) for the real trajectories.

As all constraints in the theories considered here are second class, there exist equations of motion expressing $\lambda(\lambda^I)$ in terms of $\Pi$, $q(\varphi^I, q^I)$

$$\lambda^\alpha=\psi^\alpha(\Pi, q) \qquad \lambda^{I\alpha}=\psi^{I\alpha}(\Pi^I, q^I)=\psi^\alpha(\Pi^I, q^I)+\Delta\psi^\alpha(\Pi^I, q^I).$$

Represent $W^{(1)}$ in the form

$$W^{(1)}(p, q, \lambda)=W_0^{(1)}(p, q)+(\lambda^\alpha-\psi^\alpha(\Pi, q))\phi_\beta^{(1)}(p, q)\kappa_\alpha^\beta(p, q, \lambda)\equiv W_0^{(1)}+\Delta W^{(1)}$$

where

$$W_0^{(1)}(p, q)=\varphi(\Pi, q)+\phi_\alpha^{(1)}(p, q)\bar{\Delta}^\alpha(\Pi, q, \psi(\Pi, q)).$$

Consider the action

$$S(p^I, q^I, \lambda^I)=\int \mathrm{d}t(p^I\dot{q}^I - H^{I(1)}(p^I, q^I, \lambda^I))$$

and make the transformation

$$\begin{aligned} &q^I=\tilde{q}-\{q, W_0^{(1)}\}|_{p, q\to\tilde{p}, \tilde{q}} \qquad p^I=\tilde{p}-\{p, W_0^{(1)}\}|_{p, q\to\tilde{p}, \tilde{q}} \\ &\lambda^{I\alpha}=\tilde{\lambda}^\alpha-\nabla_\lambda^\alpha(\tilde{p}, \tilde{q}, \tilde{\lambda})-\{(\lambda^\beta-\psi^\beta)\kappa_\beta^\alpha, H^{(1)}\}|_{p, q, \lambda\to\tilde{p}, \tilde{q}, \tilde{\lambda}} \end{aligned} \tag{3.8}$$

in it.

Then we have

$$S=\int \mathrm{d}t(\tilde{p}\dot{\tilde{q}}-H^{(1)}(\tilde{p}, \tilde{q}, \tilde{\lambda})-A(\tilde{p}, \tilde{q}, \tilde{\lambda})).$$

The equations of motion have the form

$$\begin{aligned} &\dot{\tilde{q}}=\{q, H^{(1)}\}|_{p, q, \lambda\to\tilde{p}, \tilde{q}, \tilde{\lambda}}+\frac{\partial A(\tilde{p}, \tilde{q}, \tilde{\lambda})}{\partial\tilde{p}} \\ &\dot{\tilde{p}}=\{p, H^{(1)}\}|_{p, q, \lambda\to\tilde{p}, \tilde{q}, \tilde{\lambda}}-\frac{\partial A(\tilde{p}, \tilde{q}, \tilde{\lambda})}{\partial\tilde{q}} \\ &\phi^{(1)}(\tilde{p}, \tilde{q})+\frac{\partial A(\tilde{p}, \tilde{q}, \tilde{\lambda})}{\partial\tilde{\lambda}}=0 \end{aligned} \tag{3.9}$$

where

$$A(p, q, \lambda)=(\lambda^\alpha-\psi^\alpha(\Pi, q))\kappa_\alpha^\beta(p, q, \lambda)\{H^{(1)}(p, q, \lambda), \phi_\beta^{(1)}(p, q)\}.$$

As the change $(p^I, q^I, \lambda^I) \rightarrow (\tilde{p}, \tilde{q}, \tilde{\lambda})$ is a point transformation, these sets either satisfy or do not satisfy the corresponding equations of motion at once. Equations of motion for $(\tilde{p}, \tilde{q}, \tilde{\lambda})$ can be obtained by substitution of the transformations (3.8) immediately in the equations of motion for $(p^I, q^I, \lambda^I)$. Consequently, these two sets of equations have the same number of generalised constraints (by generalised constraints we mean a set of ordinary constraints and equations $\lambda = \psi(\Pi, q)$). That is why the same sets of initial data are necessary for both systems. Just the same set of initial data is required also for equations of motion of variables $(p, q, \lambda)$. These equations coincide with (3.9) when $A = 0$. With the given initial data the solution of (3.9) is unique. But we know one solution. It is the solution of (3.9) when $A = 0$ because $A$ is quadratic with respect to such equations and the total time derivative of $A$ on these equations becomes zero. So the equations of motion for $(\tilde{p}, \tilde{q}, \tilde{\lambda})$ and $(p, q, \lambda)$ are, in fact, equivalent.

Consider the transformation (3.8) in more detail. The reverse transformation has the form

$$\begin{aligned} \tilde{q} &= q^I + \{q, W_0^{(1)}\}|_{p, q \rightarrow p^I, q^I} \\ \tilde{p} &= p^I + \{p, W_0^{(1)}\}|_{p, q \rightarrow p^I, q^I} \\ \tilde{\lambda} &= \lambda^I + \nabla_\lambda(p^I, q^I, \lambda^I) + \{(\lambda - \psi)\kappa, H^{I(1)}\}_{p, q, \lambda \rightarrow p^I, q^I, \lambda^I}. \end{aligned} \tag{3.10}$$

It is easy to see that on the real trajectories (3.10) coincides with (3.2), (3.3), (3.7) which, in turn, coincide with (2.1) in the $p$, $q$ sector (see (3.5) and (3.6)). The transformation $q^I = q^I(q)$ can be taken in the form (3.3) (by substituting $p = \partial L/\partial \dot{q}$, $\lambda = \dot{x}$). On the real trajectories it is inverse to (2.1). Thus we see that for the dynamics with the second-class constraints the change (2.1) (with the additional reastrictions stated above) connects physically equivalent theories and is also invertible on the real trajectories.

## 4. Non-point transformations in the constrained Hamiltonian formalism

Here we shall solve a problem which is inverse to that of section 3. We shall try to find a correspondence to the Hamilton variables transformation in the Lagrangian formalism.

Consider a theory with constraints described in the Hamiltonain formalism by the action

$$S(p, q, \lambda) = \int \mathrm{d}t(p\dot{q} - H^{(1)}(p, q, \lambda))$$

where the Hamiltonian $H^{(1)}$

$$H^{(1)}(p, q, \lambda) = H(\Pi, q) + \lambda\phi^{(1)}(p, q)$$

is constructed with regard to the Lagrangian $L(q, \dot{q})$ (see (3.1)), and $\phi^{(1)} = \pi - f(\Pi, q)$ are the primary constraints.

Let us make in $S(p, q, \lambda)$ the change of variables which is canonical in the $p$, $q$ sector but dependent, perhaps, on $\lambda$ (the generating function $W^{(1)}$ can depend on $\lambda$)

$$\begin{aligned} q &= q^I + \{q, W^{(1)}\}|_{p, q, \lambda \rightarrow p^I, q^I, \lambda^I} \\ p &= p^I + \{p, W^{(1)}\}|_{p, q, \lambda \rightarrow p^I, q^I, \lambda^I} \\ \lambda &= \lambda^I + \delta\lambda. \end{aligned} \tag{4.1}$$

The transformed action has the form (we omit the total time derivative)

$$S=\int \mathrm{d}t\left(p^I\dot{q}^I-H^{(1)}(p(p^I,q^I,\lambda^I),q(p^I,q^I,\lambda^I),\lambda^I)-\delta\lambda\phi^{(1)}(p^I,q^I)+\frac{\partial W^{(1)}}{\partial\lambda^I}\dot{\lambda}^I\right). \tag{4.2}$$

The first restriction on the change (4.1) is the absence of the time derivative of $\lambda^I$ in (4.2) (i.e. we demand that $\lambda^I$ be the Lagrange multipliers, as before). This condition yields

$$W^{(1)}=\varphi(\Pi,q)+\phi_\alpha^{(1)}(p,q)\bar{\Delta}^\alpha(p,q,\lambda) \qquad \delta\lambda=\dot{\lambda}\frac{\partial\bar{\Delta}}{\partial\lambda}+\delta_1\lambda$$

where $\delta_1\lambda$ is independent on the time derivative. The second restriction is connected with the following assumption:

$$\frac{\delta S(p^I,q^I,\lambda^I)}{\delta\lambda^{I\alpha}}=\kappa_\alpha^\beta(p^I,q^I,\lambda^I)\phi_\beta^{I(1)}(p^I,q^I)$$

where $\kappa_\alpha^\beta$ is a non-singular matrix, and the functions $\phi_\alpha^{I(1)}(p^I,q^I)$ are independent of $\lambda$. The demand of (4.1) and the first restriction correspond to the requirement that the higher derivative be absent in $L^I$ for the Lagrangian formalism. The second requirement secures the preservation of the primary constraints number. Without loss of generality one can consider that

$$\phi_\alpha^{I(1)}(p^I,q^I)=\pi_\alpha^I-f_\alpha^I(\Pi^I,q^I).$$

It is easy to show that

$$\begin{aligned}H^{I(1)}(p^I,q^I,\lambda^I)&=H^{(1)}(p,q,\lambda)-\delta_1\lambda^\alpha\phi_\alpha^{I(1)}(p^I,q^I)\\&=H^I(\Pi^I,q^I)+\phi_\alpha^{I(1)}(p^I,q^I)(\kappa^\alpha(p^I,q^I,\lambda^I)+\lambda^{I\alpha})\end{aligned}$$

and on real trajectories $\lambda^{I\alpha}=\dot{x}^{I\alpha}$.

Let us consider the Lagrangian $L^I(q^I,\dot{q}^I)$ corresponding to the Hamiltonian $H^{I(1)}$:

$$\begin{aligned}L^I(q^I,\dot{q}^I)&=(p^I\dot{q}^I-H^{I(1)}(p^I,q^I,\lambda^I))|_{\dot{q}^I=\partial H^{I(2)}/\partial p^I,\ \phi^{I(1)}=0}\\&=L(q^I,\dot{q}^I)+\Delta L(q^I,\dot{q}^I).\end{aligned}$$

Using the relation $\Delta L=-\Delta H^{(1)}$ we obtain

$$\begin{aligned}\Delta L(q,\dot{q})&=\left(\dot{q}\frac{\partial W^{(1)}}{\partial q}+\frac{\partial L}{\partial q}\frac{\partial W^{(1)}}{\partial p}\right)\bigg|_{\dot{q}=\partial H^{I(1)}/\partial p,\ \phi^{I(1)}=0}\\&=\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial\dot{q}}\delta\dot{q}+\frac{\mathrm{d}}{\mathrm{d}t}\left(W^{(1)}-p\frac{\partial W^{(1)}}{\partial p}\right)\bigg|_{\dot{q}=\partial H^{I(1)}/\partial p,\ \phi^{I(1)}=0}\end{aligned}$$

or, finally,

$$L(q,\dot{q})=L^I(q^I,\dot{q}^I)+\frac{dF}{dt}$$

$$q=q^I+\left(\frac{\partial\varphi}{\partial p}+\frac{\partial\phi^{I(1)}}{\partial p}\bar{\Delta}\right)\bigg|_{\dot{q}^I=\partial H^{I(1)}/\partial p^I,\ \phi^{I(1)}=0}\equiv q^I+\Delta$$

$$F=\left(\frac{\partial L}{\partial\dot{q}}\Delta-\varphi\right)\bigg|_{\dot{q}^I=\partial H^{I(1)}/\partial p^I,\ \phi^{I(1)}=0}$$

Thus, we have shown that if two Hamilton theories are connected by the above-mentioned change of variables, then the corresponding Lagrangians are also connected by the change of variables, and that the highest time derivative appears only in the form of the total derivative.

Note, finally, that the given proof is true for considered Hamilton theories both with second-class constraints and with first-class ones.

## 5. Conclusions

We have studied the properties of non-point transformations for constrained theories in the Lagrangian and Hamiltonian formalisms. It was shown that the non-point transformation $q \to q(q^I, \dot{q}^I)$ for the constrained Lagrange theory, also satisfying additional restrictions pointed out above, connects two physically equivalent theories and is reversible on real trajectories. The Hamiltonians of these two theories are also connected by the change $p, q, \lambda \to p^I, q^I, \lambda^I$ which, in the $p, q$ sector, is canonical (but dependent on $\lambda$) and in the $q$ sector coincides with the starting one (see (3.5)). Besides, there exists a transformation which is strictly canonical in the canonical variables sector, and has the effect of transforming real trajectories of the starting theory into those of the transformed theory. It is reversible on real trajectories, coincides in the $q$ sector with the coordinate change in the Lagrange theory and is self-consistent in the sense that on real trajectories the transformation of the Lagrange multipliers $\lambda^\alpha$ coincides with the transformation of velocities $\mathrm{d}x^\alpha/\mathrm{d}t$.

The results given in section 3 enable one to state the following important fact. Let us consider the total set of constraints $\phi_m(p, q)$ of the theory $(p, q; H^{(1)})$ and make the canonical change (3.10) in them. As this change transforms real trajectories into real ones, we obtain that the functions $\varphi_m(p^I, q^I) = \phi_m(p(p^I, q^I), q(p^I, q^I))$ become zero on real trajectories of the theory $(p^I, q^I; H^{I(1)})$. The functions $\varphi_m(p^I, q^I)$ do not depend on $\dot{q}^I$, $\dot{p}^I$ and $\dot{\lambda}^I$. Hence, they are constraints of the theory $(p^I, q^I; H^{I(1)})$. For the non-singular change, the number of independent functions $\varphi$ is equal to the number of independent functions $\phi_m$, amd then $\varphi_m$ is the total set of constraints $\phi^I_m$ of the theory $(p^I, q^I; H^{I(1)})$

$$\phi^I_m(p^I, q^I) = \phi_m(p, q)|_{p=p(p^I, q^I),\, q=q(p^I, q^I)}.$$

Note that the analogue of this relation for primary constraints is not fulfilled: the change (3.10) transforms primary constraints of the theory $(p, q; H^{(1)})$, in the general case, to a combination of primary and secondary constraints of the theory $(p^I, q^I; H^{I(1)})$.

## Acknowledgment

We wish to acknowledge useful discussions with B L Voronov and M A Vasiliev.

## References

Gantmacher F R 1966 *Lectures on Analytical Mechanics* (Moscow: Nauka)
Gitman D M and Tyutin I V 1986 *Canonical Quantization of Fields with Constraints* (Moscow: Nauka)
Goldstein H 1957 *Classical Mechanics* (Reading, MA: Addison-Wesley)
Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science, Yeshiva University)